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LETTER TO THE EDITOR

On relativistic quantum theory for particles with spin $\frac{1}{2}$†

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Abstract. We study a family of Hilbert spaces with positive definite invariant scalar product for the quantum mechanical description of relativistic spin-$\frac{1}{2}$ particles. The Hermitian and anti-Hermitian parts of the Dirac operator $\gamma^\mu p_\mu$ are related to energy and helicity. Replacing p^μ by $p^\mu - eA^\mu$, the difference between the squares of these operators provides a minimal coupling evolution operator, similar to the second-order Dirac operator, for which $F^{\mu\nu}$ is coupled to a covariant form of the Pauli spin matrices which generate SU(2) in a space-like surface. No Dirac sea is required for the consistency of the theory.

In this letter, we shall study some aspects of a manifestly relativistically covariant quantum theory (Horwitz and Piron 1973, Horwitz and Lavie 1982 and references listed there; see, in particular, Stueckelberg 1941 and 1942) for spin-$\frac{1}{2}$ particles (Horwitz *et al* 1975; Piron and Reuse 1978; some related ideas occur in van Dam and Biedenharn (1976, § III)). The wavefunctions $\psi_\tau(x)$ are defined on the manifold R^4 of space and time; they are elements of $L^2(R^4; \mathrm{d}^4x)$ (vector-valued for non-zero spin), and $|\psi_\tau(x)|^2$ is the probability density for the occurrence of an *event* at the point $x^\mu \in R^4$ at a given value of the invariant historical evolution parameter τ‡.

In the case of a particle with spin, the components of $\psi_\tau(x)$ must transform as a representation of the Lorentz group; since the norm must be invariant, the representation must be unitary. If we were to use infinite-dimensional representations containing all spins, we would reach a physical contradiction. One therefore turns to the induced representation of Wigner.

Wigner's representation corresponds, in the spin-$\frac{1}{2}$ case, to 2×2 complex unitary matrices parametrised by the particle momentum, $D(\Lambda, p)_{\sigma\sigma'}$. Since the canonical variables x^μ, p^μ satisfy (we take the metric signature to be $(-,+,+,+)$)

$$[x^\mu, p^\nu] = \mathrm{i}g^{\mu\nu} \tag{1}$$

the expectation value of x^μ, for example, would not be covariant. Hence, one uses a representation of this type induced on the little group of a time-like vector $n^\mu (n^\mu n_\mu = -1)$ which commutes with x^μ and p^ν. The (two-component spin-$\frac{1}{2}$) wavefunction then transforms as (Horwitz *et al* 1975, Piron and Reuse 1978)

$$\hat{\psi}'_{\tau n\sigma}(x) = D(\Lambda, n)_{\sigma\sigma'}\hat{\psi}_{\tau,\Lambda^{-1}n,\sigma'}(\Lambda^{-1}x). \tag{2}$$

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‡ In practice, measurements are not carried out at a fixed τ. The electromagnetic field, for example, in a semiclassical treatment (Horwitz and Lavie 1982), is generated by a conserved current which is given by an integral over all τ (i.e. associated with the entire world lines). Hence, a detector which is activated by electromagnetic interaction is sensitive to the space–time configuration of the system, but not to the values of τ (see Arshansky *et al* 1982 for a discussion of this point).

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Here

$$D(\Lambda, n) = L^{-1}(n)\Lambda L(\Lambda^{-1}n) \tag{3}$$

where $L(n)$, Λ are SL(2, C) matrices which satisfy

$$\Lambda^{+}\sigma^{\mu}n_{\mu}\Lambda = \sigma^{\mu}(\Lambda^{-1}n)_{\mu}, \tag{4}$$

and $\sigma^{\mu} = (1, \boldsymbol{\sigma})$; $L(n)$ corresponds to a transformation bringing $(\pm 1, 0, 0, 0)$ to n^{μ}. The conjugate representation, with matrices $\underset{\sim}{\Lambda}$, is also defined by (4), but with $\underset{\sim}{\sigma}^{\mu} = (1, -\boldsymbol{\sigma})$. Since $\underset{\sim}{\Lambda}\Lambda^{+} = 1$, operators linear in σ^{μ} or $\underset{\sim}{\sigma}^{\mu}$ connect these representations, and hence we define the norm for the spin-$\frac{1}{2}$ system, in the Hilbert space $\mathcal{H}_n$, as (in the following, we differ from the treatment given in Piron and Reuse (1978))

$$N = \int \mathrm{d}^4x \; (|\hat{\psi}_{\tau n}(x)|^2 + |\hat{\varphi}_{\tau n}(x)|^2) \tag{5}$$

where $\hat{\varphi}_{\tau n}$ transforms with $\underset{\sim}{D}$. from (2) and (3), it follows that $L(n)\hat{\psi}_{\tau n}$ transforms with Λ, and $\underset{\sim}{L}(n)\hat{\varphi}_{\tau n}$ with $\underset{\sim}{\Lambda}$; making this replacement in (5), and using the fact, obtained from (4), that $L(n)^{+-1}L(n)^{-1} = \mp\sigma^{\mu}n_{\mu}$ and $\underset{\sim}{L}(n)^{+-1}\underset{\sim}{L}(n)^{-1} = \mp\underset{\sim}{\sigma}^{\mu}n_{\mu}$, one finds that (we use the representation for γ^{μ} given by Bjorken and Drell (1964))

$$N = \mp \int \mathrm{d}^4x \; \bar{\psi}_{\tau n}(x)\gamma \cdot n\psi_{\tau n}(x) \tag{6}$$

where $\gamma \cdot n \equiv \gamma^{\mu}n_{\mu}$ $((\gamma \cdot n)^2 =))$,

$$\psi_{\tau n}(x) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} L(n)\hat{\psi}_{\tau n}(x) \\ \underset{\sim}{L}(n)\hat{\varphi}_{\tau n}(x) \end{pmatrix} \tag{7}$$

and $\mp$ corresponds to n^{μ} in the positive or negative light cone. The wavefunctions defined by (7) transform as

$$\psi'_{\tau n}(x) = S(\Lambda)\psi_{\tau\Lambda^{-1}n}(\Lambda^{-1}x), \tag{8}$$

and $S(\Lambda)$ is generated in the usual way by $\Sigma^{\mu\nu} = \frac{1}{4}\mathrm{i}[\gamma^{\mu}, \gamma^{\nu}]$.

The Dirac operator $\gamma \cdot p$ is not Hermitian in the (invariant) scalar product associated with (6). We therefore consider the Hermitian and anti-Hermitian parts

$$\begin{aligned} K_{\mathrm{L}} &= \tfrac{1}{2}(\gamma \cdot p + \gamma \cdot n\,\gamma \cdot p\,\gamma \cdot n) = -(p \cdot n)(\gamma \cdot n), \\ K_{\mathrm{T}} &= \tfrac{1}{2}\gamma^5(\gamma \cdot p - \gamma \cdot n\,\gamma \cdot p\,\gamma \cdot n) = -2\mathrm{i}\gamma^5(p \cdot K)(\gamma \cdot n), \end{aligned} \tag{9}$$

where $K^{\mu} = \Sigma^{\mu\nu}n_{\nu}$, and we have introduced the factor γ^5 in the second of (9) so that K_{T} is Hermitian and commutes with K_{L}. Since

$$K_{\mathrm{L}}^2 = (p \cdot n)^2 \qquad \text{and} \qquad K_{\mathrm{T}}^2 = p^2 + (p \cdot n)^2, \tag{10}$$

we may consider $K_{\mathrm{T}}^2 - K_{\mathrm{L}}^2 = p^2$ as the operator analogous to the second-order mass eigenvalue condition for the free Dirac equation. For the equation of evolution (Horwitz and Piron 1973)

$$\mathrm{i}\partial\psi_{\tau n}/\partial\tau = K\psi_{\tau n} \tag{11}$$

we choose the free dynamical evolution operator to be

$$K_0 = (1/2M)(K_{\mathrm{T}}^2 - K_{\mathrm{L}}^2). \tag{12}$$

Since (in the absence of electromagnetism), K_{L} and K_{T} commute with each other and

K_0, and $\gamma \cdot n$ commutes with these operators as well ($\{\gamma \cdot n, K^n\} = 0$), the free solutions can be decomposed according to the projections

$$P_\pm = \tfrac{1}{2}(1 \mp \gamma \cdot n), \qquad P_{E\pm} = \tfrac{1}{2}\left(1 \mp \frac{p \cdot n}{|p \cdot n|}\right), \qquad P_{n\pm} = \tfrac{1}{2}\left(1 \pm \frac{2\mathrm{i}\gamma^5 K \cdot p}{[p^2 + (p \cdot n)^2]^{1/2}}\right). \quad (13)$$

In the special frame for which $n^\mu = (1, 0, 0, 0)$, $P_{n\pm}$ is seen to correspond to helicity:

$$2\mathrm{i}\gamma^5 K \cdot p / [p^2 + (p \cdot n)^2]^{1/2} \to \boldsymbol{\sigma} \cdot \boldsymbol{p} / |\boldsymbol{p}|. \quad (14)$$

In the presence of electromagnetic interaction, we replace p by $p - eA$ in K_{L} and K_{T}. The operator (12) then becomes ($\gamma \cdot n$ commutes with this K as well; note that γ^5 also commutes with K and hence a chiral decomposition is also possible)

$$K = (p - eA)^2/2M + (e/2M)\Sigma_n^{\mu\nu} F_{\mu\nu}(x) \quad (15)$$

where†

$$\Sigma_n^{\mu\nu} = \Sigma^{\mu\nu} + K^\mu n^\nu - K^\nu n^\mu. \quad (16)$$

The quantities K^μ, $\Sigma^{\mu\nu}$ satisfy

$$[K^\mu, K^\nu] = -\mathrm{i}\Sigma_n^{\mu\nu},$$
$$[\Sigma_n^{\mu\nu}, K^\lambda] = -\mathrm{i}[(g^{\nu\lambda} + n^\nu n^\lambda)K^\mu - (g^{\mu\lambda} + n^\mu n^\lambda)K^\nu], \quad (17)$$

$$[\Sigma_n^{\mu\nu}, \Sigma_n^{\lambda\sigma}] = -\mathrm{i}[(g^{\nu\lambda} + n^\nu n^\lambda)\Sigma_n^{\mu\sigma} + (g^{\sigma\mu} + n^\sigma n^\mu)\Sigma_n^{\lambda\nu}$$
$$- (g^{\mu\lambda} + n^\mu n^\lambda)\Sigma_n^{\nu\sigma} + (g^{\sigma\nu} + n^\sigma n^\nu)\Sigma_n^{\lambda\mu}]. \quad (18)$$

Since $K^\mu n_\mu = n^\mu \Sigma_n^{\mu\nu} = 0$, there are only three independent K^μ and three $\Sigma_n^{\mu\nu}$. The $\Sigma_n^{\mu\nu}$ (see remark after equation (19)) are a covariant form of the Pauli matrices, and (18) is the corresponding form of SU(2) in the space-like hypersurface orthogonal to n^μ. The three independent K^μ correspond to the non-compact part of the algebra. The covariance of the theory follows from

$$S^{-1}(\Lambda)\Sigma_{\Lambda n}^{\mu\nu} S(\Lambda)\Lambda^\lambda_\mu \Lambda^\sigma_\nu = \Sigma_n^{\lambda\sigma}. \quad (19)$$

In the special frame for which $n^\mu = (1, 0, 0, 0)$, Σ_n^{ij} becomes $\frac{1}{2}\sigma^k$ (i, j, k cyclic) and Σ_n^{0j} goes to zero. In this frame (which we understand as the frame of the filter preparing the state), there is no electric interaction with the spin in the minimal coupling evolution operator (15). We remark that a spin coupling which becomes pure electric in the special frame is generated by (for $p \to p - eA$)

$$\mathrm{i}[K_{\mathrm{T}}, K_{\mathrm{L}}] = -\mathrm{i}e\gamma^5(K^\mu n^\nu - K^\nu n^\mu)F_{\mu\nu}. \quad (20)$$

The discrete symmetries act on the wavefunctions as follows:

$$\psi_{\tau n}^C(x) = C\gamma^0 \psi_{-\tau,n}^*(x) \qquad (C = \mathrm{i}\gamma_2\gamma^0), \quad (21)$$
$$\psi_{\tau n}^P(x) = \gamma^0 \psi_{\tau,-\boldsymbol{n},n^0}(-\boldsymbol{x}, t), \quad (22)$$
$$\psi_{\tau n}^T(x) = \mathrm{i}\gamma^1\gamma^3 \psi_{-\tau,\boldsymbol{n},-n^0}^*(\boldsymbol{x}, -t), \quad (23)$$
$$\psi_{\tau n}^{CPT}(x) = \mathrm{i}\gamma^5 \psi_{\tau,-n}(-\boldsymbol{x}, -t). \quad (24)$$

The *CPT* conjugate wavefunction, according to its evolution in τ, moves backwards

† Internal angular momentum variables of the type $\Sigma_n^{\mu\nu}$, K^μ have been used by Todorov (1981), but using p^μ in place of n^μ (in Todorov's work, $\Sigma_p^{\mu\nu}$ is equivalent to $\Sigma^{\mu\nu}$ on the constraint hypersurface).

in space–time relative to the motion of $\psi_{\tau n}$. If $\psi_{\tau n}$ contains only $E>0$ components, it moves forward in t (Horwitz and Piron 1973) and corresponds to the system observed in the laboratory. If it contains only $E<0$ components, the wave packet moves backward in t. It is then the *CPT* conjugate which corresponds to the observed system moving forward in time, with charge $-e$. No Dirac sea is required for the consistency of the theory (unbounded transitions to $E<0$ are prevented by the conservation of K).

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